Journal of Functional Analysis **182**, 1–15 (2001)
doi:10.1006/jfan.2000.3717, available online at http://www.idealibrary.com on IDEAL®

The Range of Vector Measures over Topological Sets and a Unified Liapunov Theorem

Daniel Wulbert

Department of Mathematics 0112, University of California, San Diego, La Jolla, California 92093
E-mail: dwulbert@ucsd.edu

Communicated by D. Sarason

Received October 13, 1999; accepted September 26, 2000

Liapunov's classsical theorem is that also $Q=\{M(s): s\in \operatorname{ext} B(L_\infty)\}$. This paper characterizes when the functions, s, in Liapunov's theorem can further be restricted to being the signs of continuous functions. That is, suppose there is a topology on X, and that Σ is the Baire sets. Let S be the collection of supports of continuous non-negative functions.

THEOREM. *The following are equivalent*:

(i) $Q=\{M(s): s\in \operatorname{ext} B(L_\infty)$, *and* $s=\operatorname{sgn} f$ *for some* $f\in C(X)\}$,

(ii) *for every* $g\in G$, $\{x: g(x)>0\}$ *and* $\{x: g(x)\leqslant 0\}$ *are in* S, *a.e.*

The theorem is proved in a general setting. If the σ-field in the general theorem is arbitrary, the theorem becomes Liapunov's Theorem. The theorem above results when the σ-field is the Baire sets. A setting with the Borel sets produces Q as the range of M over extreme functions that are both lower semi-continuous a.e. and upper semi-continuous a.e.

Key Words: nonatomic Baire measures; vector measures; convex sets; supports of continuous functions; extreme points.

1. INTRODUCTION

Liapunov's Theorem [6] is equivalent to saying $Q^+=\{M(I_K): K\in\Sigma\}$ is convex. Render and Stroetmann [9] initiated research on topological versions of Liapunov's Theorem by proving sufficient conditions that Q^+ be realized when the sets K are restricted to the open sets of a topological space X, (and where Σ is the Borel sets). Kellerer [5] provided the complete characterization by showing that $Q^+=\{M(I_U): U \text{ open}\}$ if and only

0022-1236/01 $35.00

if for every $g \in G$, $\text{supp}(g^+)$ and $[\text{supp}(g^+)]^c$ are both open sets, a.e. (that is, each differs from an open set by a set of measure zero).

This paper has two puposes. The first is to prove topological versions of Liapunov's theorem that realize Q as the range of M over smaller fundamental classes of extreme functions. These will include the class of extreme functions that are the sign of a continuous function, and the class of extreme functions that are both lower semi-continuous a.e. and upper semi-continuous a.e.

The second purpose is to present a general setting from which these different theorems (including Liapunov's original theorem) are derived as special cases by specifing the σ-algebras involved.

Comments

(i) An elegant direct proof of Liapunov's Theorem is in [7].

(ii) For the characterization involving continuous functions, we require that Σ be the Baire sets (i.e., the σ-algebra generated by the supports of non-negative continuous functions). Often, with regularity conditions assumed in analysis settings, the Borel sets are also Baire sets. In general they are distinct. For example, Hewitt [4] found a regular space (i.e., K closed and $x \notin K$ implies there are open disjoint sets U and V with $x \in U$ and $K \subseteq V$) on which every continuous function is constant. Note that a Baire measure on such a space would be atomic.

(iii) The paper assumes no regularity conditions on the topology or on the associated measure space.

(iv) Various proofs and applications of the classical Liapunov theorem are found in Refs. [1, 3, 6, 8, 11, 12].

2. NOTATION AND DEFINITIONS

Measure Spaces. Throughout the paper (X, Σ, μ) will be a measure space.

For $U, K \in \Sigma$, we say that *U splits K* if $\mu(K) > \mu(K \cap U) > 0$. A set $K \in \Sigma$ is an *atom* of μ if $\mu(K) > 0$, and no U in Σ splits K. (X, Σ, μ) is *nonatomic* if no set in Σ is an atom with respect to μ.

We use a.e. as an abbreviation for the term "almost everywhere." For example, we will write U is open a.e. to mean there is an open set V such that $\mu((U-V) \cup (V-U)) = 0$. Other authors call this μ-equivalent to an open set.

We will sometimes write L_1 for $L_1(X, \Sigma, \mu)$. If there is also a topology on X, a set in the smallest σ-algebra containing the open sets is a *Borel* set.

The σ-algebra generated by the supports of the non-negative, continuous real functions is the *Baire* sets.

Functions. The linear span of a collection of functions, F, is span F.

The sign of a function f is $\operatorname{sgn} f(x) := \{1$ if $f(x) > 0$; -1 if $f(x) < 0$; and 0 if $f(x) = 0\}$. We will use f^+ and f^- to be $\max\{f(x), 0\}$ and $-\min\{f(x), 0\}$, respectively. The *characteristic* or *indicator* function of a set $A \subseteq X$ is $I_A(x) := \{1$ if $x \in A$; and 0 if $x \notin A\}$.

Spaces of Functions. The essentially bounded functions on X are written as L_∞. Its closed unit ball is $B(L_\infty)$, and the extreme points of the unit ball is ext $B(L_\infty) = \{f \in L_\infty; |f| = 1$, a.e.$\}$. For a collection of functions H, we use $H^+ = \{h \in H : h \geqslant 0\}$.

The continuous functions are written $C(X)$, and $C^+(X) = \{f \in C(X) : 0 \leqslant f\}$. If $G \subseteq L_1$, then $G^\perp = \{f \in L_\infty : \int_X fg \, d\mu = 0$ for all $g \in G\}$. f is upper semi-continuous a.e. if for all real numbers a, $f^{-1}(a, \infty)$ is open a.e.

Sets. For sets K and U, we use $K - U$ to represent $\{x \in K : x \notin U\}$, and the *complement* of K is $K^c := X - K$.

Let f be a real-valued function defined on X. The *support* of f is $\operatorname{supp}(f) := \{x : f(x) \neq 0\}$. We define the *support* of G to be $\operatorname{supp} G := \bigcup_{i=1}^n \operatorname{supp}(g_i)$. The *zero set* of f is $Zf = Z(f) := \{x \in X : f(x) = 0\}$. We may also use the notation such as $\{f > a\}$ for $\{x : f(x) > a\}$ when it is easier to read. For example, $(\operatorname{supp} g^+)^c$ is more easily recognized as $\{g \leqslant 0\}$.

Reserved Notation for This Paper. For $G = \operatorname{span}\{g_1, g_2, ..., g_n\} \subseteq L_1$, and an $f \in L_\infty$ we reserve the notation

$$M(f) = \left(\int_X fg_1 \, d\mu, \int_X fg_2 \, d\mu, ..., \int_X fg_n \, d\mu \right),$$

$$Q = \{M(h) : h \in L_\infty, -1 \leqslant h \leqslant 1\},$$

and

$$Q^+ = \{M(h) : h \in L_\infty, 0 \leqslant h \leqslant 1\}.$$

DEFINITION. Let $S \subseteq \Sigma$. We will say that a function f is S-*supported* if the three sets $\operatorname{supp} f^+$, Zf, and $\operatorname{supp} f^-$ are in S a.e. (that is, each differs from a set in S by a set of measure zero). We say that a collection of functions H is S-*supported* if each member of H is S-supported. We alert the reader, that this definition of S-supported is not a standard definition.

We list below variations of this definition that we use later.

LEMMA 2.1. *Let S be closed under finite unions and finite intersections. Let H be a linear subspace of functions. The following are equivalent*:

(1) *H is S-supported,*

(2) *for each $h \in H$ the sets $\{h > 0\}$ and $\{h \leqslant 0\}$ are in S, a.e.*

(3) *for each $h \in H$, $\{h > 0\}$ and Zh are in S a.e.*

LEMMA 2.2. *If $S = \{\operatorname{supp} f : f \in C^+(X)\}$, then $s \in \operatorname{ext} B(L_\infty)$ is S-supported if and only if there is a continuous function f such that $s = \operatorname{sgn} f$.*

3. THE SETTINGS

In this section we isolate the properties used in the proof of the unified Liapunov theorem, Theorem 4.1, below.

Let $\mathscr{L}$ be a convex cone of functions in L_∞, and let $S = \{\operatorname{supp} f : f \in \mathscr{L}^+\}$ generate Σ. We hypothesize:

(i) S is closed under finite unions and intersections; and if $\{N_i\}$ and $\{P_i\}$ are disjoint sequences of nested sets in S (i.e., $N_i \subseteq N_{i+1}$. a.e. and $P_i \subseteq P_{i+1}$. a.e.) such that the disjoint (a.e.) sets $N = \bigcup_{i=1}^\infty N_i$ and $P = \bigcup_{i=1}^\infty P_i$ union (a.e.) to X; then N and P are in S.

(ii) If S_1, S_2, and S_3 are pairwise disjoint a.e. subsets in S that union to X a.e., and f_1, f_2, and f_3 are functions in $\mathscr{L}$, then there is an f in $\mathscr{L}$ such that $\operatorname{sgn} f = \operatorname{sgn} f_i$ on S_i.

(iii) There is an $f \in \mathscr{L}^+$, such that $\mu(f^{-1}(a)) = 0$ for all real a and such that

$$\{\mu(\{x : f(x) > a\}) : 0 \leqslant a \leqslant \|f\|_\infty\} = [0, \mu(X)].$$

(iv) If H is a finite dimensional subspace of $L_1(X, \Sigma, \mu)$, there is a $q \in \mathscr{L}$, such that, $\operatorname{sgn} q \in H^\perp$, and $\mu(Z(q)) = 0$.

LEMMA 3.1. *The following satisfy the conditions above*:

(a) *$\mathscr{L} = L_\infty$, and $S = \Sigma$.*

(b) *Σ is the Baire sets in X; $\mathscr{L} = C(X)$; and S consists of the supports (a.e.) of non-negative, bounded, continuous functions.*

(c) *S is the subsets of X that are both open a.e. and closed a.e.; Σ is the σ-field generated by S; and $\mathscr{L} = \{f \in L_\infty : f^{-1}(a, \infty) \in S\}$.*

(d) *Suppose that the subsets of X that are both open a.e. and closed a.e. form a basis for the topology on X (e.g., Lebesgue measure on $\mathbf{R}^n$). Then* *let S be the open sets (a.e.); Σ the Borel sets; and $\mathscr{L}$ the lower semi-continuous (a.e.) functions.*

Proof. Condition (i) is readily verifiable.

To show that L_∞ satisfies condition (ii) let $f = f_i$ on S_i. In $C(X)$ let s_i be a non-negative, bounded continuous function whose support is S_i.

Condition (ii) is satisfied by putting $f = s_1 f_1 + s_2 f_2 + s_3 f_3$. If f is a function from either setting (c) or (d), then $f = \operatorname{sgn} f_i$ on S_i, fulfills condition (ii).

Condition (iii) is Theorem 5.2 and Lemma 6.2 in Wulbert [14].

Condition (iv) follows from Lemma 5.1 in Wulbert [13]. That lemma states that if Q is an $n+1$-dimensional subspace of bounded measurable functions such that $\mu(Zq) = 0$ for all q in Q; then, If H is $n-1$-dimensional, there is a $q \in Q$ for which $\operatorname{sgn} q \in H^\perp$. In the three settings (a), (b), and (c) of the current lemma, the function f satisfying condition (iv) has the property that $\mu(f^{-1}(a)) = 0$ for all real numbers a. Hence, $Q = \{p \circ f : p$ a polynomial of degree $\leqslant n\}$, satisfies the referred to lemma. Hence there is a $q \in Q$ such that $\operatorname{sgn} q \in H^\perp$. The $\mathscr{L}$ in settings (a), (b), and (c) form an algebra. So, q is also in that space, satisfies condition (iii). Setting (d) for the lower semi-continuous (a.e.) functions follows from setting (c), since the functions in (c) are all lower semi-continous a.e. ∎

Comments. Condition (i) will be used to show that a particular L_1 limit of S-supported functions is also S-supported. The condition is, of course, less restictive then assuming that S is closed under countable unions. This less restrictive condition was used to include the sets that are both open a.e. and closed a.e. (setting (c) in the lemma above).

If $Q = \{M(\operatorname{sgn} f) : f \in \mathscr{L}, \mu(Z_f) = 0\}$ then of course (since $0 \in Q$) there is a $q \in \mathscr{L}$ such that $\operatorname{sgn} q \in G^\perp$, and $\mu(Z(q)) = 0$. Condition (iv) requires the conclusion holds for all finite dimensional subspaces $H \subseteq L_1$. The setting that did not readily satisfy condition (iv) is: $\mathscr{L}$ the lower semi-continuous functions, S the open sets a.e., and Σ the Borel sets. This is the setting of Kellerer's Theorem. Our proof that the settings (a), (b), and (c) of the lemma satisfied condition (iv) used the fact that in those settings $\mathscr{L}$ is an algebra.

4. A UNIFIED LIAPUNOV THEOREM

LEMMA 4.1. *Let $\mathscr{L}$ and S be as hypothesized in Section* 3. *If G is S-supported, then $Q = \{M(\operatorname{sgn} f) : f \in \mathscr{L}, \mu(Z_f) = 0\}$.*

Proof. For simplicity, we assume that $\mu(X) = 1$. The proof will be by induction on the dimension of G. The case for $G = \operatorname{span}\{g_1\}$ will follow from hypothesized condition (iii) of Section 3. We apply the condition separately to each of the sets $\{g_1>0\}$, and $\{g_1<0\}$. By hypothesis both sets are in S. We start the proof of the one dimensional case by showing that for $-\|g_1^+\|_\infty \leqslant a_1 \leqslant \|g_1^+\|_\infty$, there is a function $f_1 \in \mathscr{L}$ such that

$$a_1=\int_{\{g_1>0\}} \operatorname{sgn}(f_1)\, g_1^+ \, d\mu.$$

To find this f_1 apply condition (iii) to the nonatomic measure, v, corresponding to $g_1^+ \, d\mu$. This gives us a non-negative, $y \in \mathscr{L}$, defined on X such that

$$\{v(\{x : y(x)>a\}) : 0 \leqslant a \leqslant \|y\|_\infty\} = [0, v(X)] = [0, \|g_1^+\|_\infty].$$

For all numbers a, $v(y^{-1}(a))=0$. Hence,

$$\left\{\int_X \operatorname{sgn}(y-a)\, g_1^+ \, d\mu; 0 \leqslant a \leqslant \|y\|_\infty\right\} = [-\|g_1^+\|_\infty, \|g_1^+\|_\infty].$$

Choose a^* so that $\int_X \operatorname{sgn}(y-a^*)\, g_1^+ \, d\mu = a_1$.

Similarly for $-\|g_1^-\|_\infty \leqslant a_2 \leqslant \|g_1^-\|_\infty$ there is an $f_2 \in \mathscr{L}$ such that,

$$a_2=\int_{\{g_1<0\}} \operatorname{sgn}(f_2)\, g_1^- \, d\mu.$$

Since $Z(g_1)$ is also in S, hypothesized condition (ii) of Section 3 provides an $f \in \mathscr{L}$ such that $\operatorname{sgn} f = \operatorname{sgn} f_1$ on $\{g_1>0\}$; $\operatorname{sgn} f = \operatorname{sgn} f_2$ on $\{g_1<0\}$; and $\operatorname{sgn} f = 1$ on $Z(g_1)$.

Now if $h \in B(L_\infty)$, let $a_1 = \int hg_1^+ \, d\mu$, and $a_2 = \int hg_1^- \, d\mu$. Then the above shows that there is an $f \in \mathscr{L}$ with $\mu(f^{-1}(0))=0$ such that $\int \operatorname{sgn} fg_1 \, d\mu = \int hg_1 \, d\mu$. Hence,

$$\left\{\int_X sg_1 \, d\mu : s \in \operatorname{ext} B(L_\infty) : s \text{ is } S\text{-supported}\right\} = [-\|g_1\|_\infty, \|g_1\|_\infty].$$

That shows that the theorem is true if the dimension of G is one.

Now we assume that the lemma is true for G having dimension up to $n-1$. Let $h \in B(L_\infty)$. We seek an $f \in \mathscr{L}$ such that $\mu(Z(f))=0$, and $M(h) = M(\operatorname{sgn} f)$. If $h \in G^\perp$, then hypotesized condition (iv) of Section 3 guarantees that such an f exists. Otherwise let

$$G_h = \left\{g \in G : \int_X gh \, d\mu = 0\right\}.$$

Let $g_h \in G$ be such that $\|g_h\|_1 = 1$, and g_h has 0 as a best approximation (in L_1) from G_h. Then there is a $w \in L_\infty$ such that:

(a) $\|w\|_\infty = 1$,

(b) $\int_X wg\, d\mu = 0$ for $g \in G_h$, and

(c) $\int_X wg_h\, d\mu = \|g_h\|_1$.

Condition (c) implies that $w = \operatorname{sgn} g_h$, a.e. on supp g_h.

We seek an s that is an S-supported function in $G_h^\perp \cap \operatorname{ext} B(L_\infty)$, such that

$$\int_X sg_h\, d\mu = \alpha = \int_X hg_h\, d\mu.$$

If such an s exists, then for all $g \in G$,

$$\int_X sg\, d\mu = \alpha = \int_X hg\, d\mu.$$

Furthermore, the hypothesized condition (ii) of Section 3 provides that s is the sign of a function in $\mathscr{L}$ and the proof would be completed.

We will inductively construct a sequence of functions $\{a_i\}$ that converges in L_1 to such an s.

We first construct a_1. We will define an upper and lower bound (u_1 and l_1) for a_1.

By hypothesis $Z(g_h) \in S$. It is possible that $\mu(Z(g_h)) = 0$. But in any case, by the induction hypothesis, applied to the space G_h (where $X = Z(g_h)$ and w plays the role of "h"), there is a $v_0 \in \mathscr{L}$, supported on $Z(g_h)$ such that

$$\int_{Z(g_h)} g \operatorname{sgn} v_0\, d\mu = \int_{Z(g_h)} gw\, d\mu \qquad \text{for all} \quad g \in G_h, \qquad \text{and}$$

$$\mu(Z(v_0) \cap Z(g_h)) = 0.$$

Put

$$u_1 = \begin{cases} \operatorname{sgn} g_h, & \text{on supp } g_h, \\ \operatorname{sgn} v_0, & \text{on } Z(g_h). \end{cases}$$

Notice that $Z(u_1) = 0$.

From the hypothesized condition (iv), there is an $l_1 \in \operatorname{ext} B(L_\infty)$ (and hence $Z(1_1) = 0$) such that l_1 is S-supported, and

$$\int_X l_1 g\, d\mu = 0, \qquad \text{for all} \quad g \in G.$$

Put

$$\alpha = \int_X hg_h \, d\mu.$$

We have that l_1 and u_1 are S-supported functions in $G_h^{\perp} \cap \operatorname{ext} B(L_\infty)$, such that

$$0 = \int_X l_1 g_h \, d\mu \leqslant \alpha = \int_X hg_h \, d\mu \leqslant \int_X u_1 g_h \, d\mu = \int_X |g_h| \, d\mu.$$

Notice that we have assumed that $\alpha > 0$. If that were not the case, we would use $-u_1$ in place of u_1 above. This would still be an S-supported function of absolute value 1 a.e. The inequalities in the line above would be reversed (as well as in the following argument), but all the subsequent logic is the same and yields the same conclusion.

We now start the construction of the general step in the inductive construction of the functions converging in L_1 to s.

Let $U_0 = \emptyset$. Suppose that we have constructed, l_i, u_i, and U_{i-1} with the following properties:

(1) $l_i, u_i \in G_h^{\perp} \cap \operatorname{ext} B(L_\infty)$, (hence $\mu(Z(u_i)) = \mu(Z(l_i)) = 0$).

(2) l_i and u_i are S-supported,

(3) $\int_X l_i g_h \, d\mu \leqslant \alpha = \int_X hg_h \, d\mu \leqslant \int_X u_i g_h \, d\mu$, and

(4) $l_i(x) = u_i(x)$ for $x \in U_{i-1}$,

(5) $U_{i-2} \subseteq U_{i-1}$ for $i > 1$, and

(6) $\mu(U_{i-1}^c) = 1/2^{i-1}$ for $i > 1$.

Put $U_i = \{x \in X : u_i(x) = l_i(x)\}$. We proceed to define l_{i+1} and u_{i+1}. Let $a_i = (u_i + l_i)/2$. Then $U_i = \operatorname{supp} a_i$ and $U_i^c = Z(a_i)$. Both U_i and U_i^c are in S. For example, $U_i^c = [\{u_i > 0\} \cap \{l_i < 0\}] \cup [\{u_i < 0\} \cap \{l_i > 0\}]$. Also $P_i = \{x \in X;\, a_i(x) = 1\}$ and $N_i = \{x \in X;\, a_i(x) = -1\}$ are disjoint sets in S that union to U_i.

From condition (iv) (applied to the restriction of μ to U_i^c, and to the space generated by G, l_i, and u_i) there is a function $v_i \in \mathscr{L}$ such that $|v_i| = 1$ on U_i^c, $\mu(Z(v_i) \cap U_i^c) = 0$, and $\int_{U_i^c} v_i g \, d\mu = 0$ for $g \in G$, or $g \in \{l_i, u_i\}$.

Put

$$m_i(x) = \begin{cases} a_i(x) & \text{for} \quad x \in U_i, \\ v_i(x) & \text{for} \quad x \in U_i^c. \end{cases}$$

Then $m_i \in G_h^{\perp} \cap \operatorname{ext} B(L_\infty)$, and

$$\int_X m_i g_h \, d\mu = \tfrac{1}{2}\left[\int_X u_i g_h \, d\mu + \int_X l_i g_h \, d\mu\right].$$

If $\int_X m_i g_h \, d\mu \leqslant \alpha \leqslant \int_X u_i g_h \, d\mu$, then put

$$u_{i+1} = u_i, \qquad \text{and} \qquad l_{i+1} = m_i.$$

Otherwise put

$$u_{i+1} = m_i, \qquad \text{and} \qquad l_{i+1} = l_i.$$

The only condition remaining to be satisfied is that $\mu(U_i^c) = 1/2^i$. We compute that

$$\frac{1}{2^i} = \frac{1}{2}\,\mu(U_{i-1}^c) = \int_{U_{i-1}^c} \frac{1}{2}\,|u_i| \, d\mu = \int_{U_{i-1}^c} \frac{1}{2}\,[u_i + l_i]\, u_i \, d\mu = \int_{U_{i-1}^c} I_{U_i} \, d\mu$$
$$= \mu(U_i \cap U_{i-1}^c).$$

The second equality follows from $|u_i| = 1$; the third from our requirement that $v_{i-1} \in \{u_{i-1}, l_{i-1}\}^{\perp}$, hence $\int_{U_{i-1}^c} l_i u_i \, d\mu = 0$; and the fourth from the fact that,

$$\frac{u_i + l_i}{2}(x) = \left.\begin{cases} u_i(x) & \text{for} \quad x \in U_i \\ 0 & \text{for} \quad x \in U_i^c \end{cases}\right\} = u_i I_{U_i}.$$

Since $U_i^c \subseteq U_{i-1}^c$,

$$\mu(U_i^c) = \mu(U_{i-1}^c) - \mu(U_i \cap U_{i-1}^c) = \frac{1}{2^{i-1}} - \frac{1}{2^i} = \frac{1}{2^i}.$$

Hence,

$$\bigcup_{i=1}^{\infty} U_i = X, \qquad \text{a.e.}$$

We have constructed l_{i+1}, u_{i+1} and U_i to satisfy the properties corresponding to (1) through (6) above. The final step is to employ those properties to construct the desired function, s, that is, (a) S-supported, (b) in $G_h^{\perp} \cap \operatorname{ext} B(L_\infty)$, and (c) $\int_X s g_h \, d\mu = \alpha = \int_X h g_h \, d\mu$.

Again let $a_i=(u_i+l_i)/2$, $P_i=\{x\in X;\ a_i(x)=1\}$, and $N_i=\{x\in X;\ a_i(x)=-1\}$. Again $P_i\subseteq P_{i+1}$ and $N_i\subseteq N_{i+1}$. Since $i\geqslant j$ implies that $a_i=1$ on P_j, and $a_i=-1$ on N_j; we conclude that a_i converges pointwise to

$$s(x)=\begin{cases} 1 & \text{for}\quad x\in \bigcup_{i=1}^{\infty} P_i=P,\\ -1 & \text{for}\quad x\in \bigcup_{i=1}^{\infty} N_i=N.\end{cases}$$

Since

$$\left[\bigcup_{i=1}^{\infty} P_i\right]\cup\left[\bigcup_{i=1}^{\infty} N_i\right]=\bigcup_{i=1}^{\infty} U_i=X, \text{ a.e.},$$

from the Dominated Convergence Theorem, $a_i p$ converges to sp with respect to L_1 for all integrable functions p. In particular, for all $g\in G$,

$$\int_X sg\,d\mu=\lim_{i\to\infty}\int_X a_i g\,d\mu.$$

Hence, $\int_X sg\,d\mu=0$ for $g\in G_h$. Also

$$\lim_{i\to\infty}\int_X a_i g_h\,d\mu=\alpha$$

since

$$\int_X l_i g_h\,d\mu\leqslant\alpha\leqslant\int_X u_i g_h\,d\mu,$$

$$\int_X l_i g_h\,d\mu\leqslant\int_X a_i g_h\,d\mu\leqslant\int_X u_i g_h\,d\mu,$$

and

$$\left|\int_X a_i g_h\,d\mu-\alpha\right|\leqslant\int_X u_i g_h\,d\mu-\int_X l_i g_h\,d\mu\leqslant\int_X |u_i-l_i|\,|g_h|\,d\mu$$

$$\leqslant\int_X 2I_{U_i^c}\,|g_h|\,d\mu.$$

From the Dominated Convergence Theorem, the last integral converges to zero. Finally we observe that from hypothesized condition (i) of Section 3, $\bigcup_{i=1}^{\infty} P_i=P$, and $\bigcup_{i=1}^{\infty} N_i=N$ are in S. Therefore, s is S-supported. ∎

COROLLARY 4.2 (Liapunov's Theorem). *Let* (X, Σ, μ) *be a finite non-atomic measure space. Then*

$$\{M(h) : h \in B(L_\infty)\} = \{M(s) : s \in \text{ext } B(L_\infty)\}.$$

Proof. This is immediate from Theorem 4.1 and Lemma 3.1 part (a). ∎

5. NECESSITY OF G BEING S-SUPPORTED

Let (X, Σ, μ) be any measure space. Let G be an n-dimensional subspace of $L_1(X, \Sigma, \mu)$. Let S be a subcollection of Σ that is closed under finite intersections and finite unions.

THEOREM 5.1. *Suppose* $\text{supp } G = X$, *a.e. If* $Q^+ = \{M(I_A) : A \in S\}$, *then* G *is* S*-supported.*

Comment. The proof consists of showing that if $g_0 \in G$, then the two sets $P_0 = \{g_0 > 0\}$ and $P_0^c = \{g_0 \leqslant 0\}$ are in S, a.e. The argument is a delicate inductive construction. The notation of lexicographic ordering would provide an alternate procedure for visualizing the construction below. For example, the sets, below, $\bigcup_{i=0}^{w} P_i$, and $P_0 \cup [\bigcup_{i=1}^{w} N_i]$ would become $\{x : (g_0, g_1, ..., g_w) \succ 0\}$ and $\{x : (g_0, -g_1, -g_2, ..., -g_w) \succ 0\}$, respectively, where we define $\{x : (h_0, h_1, ..., h_w) \succ 0\}$ if and only if (i) there is some j in $\{0, 1, ..., w\}$ such that $h_j(x) \neq 0$, and (ii) $h_{\min\{j : h_j(x) \neq 0\}} > 0$. A lexicographic procedure such as this was used by Kellerer [5, p. 212].

Proof. Let $g_0 \in G$. We will show that $P_0 = \{g_0 > 0\}$ and $P_0^c = \{g_0 \leqslant 0\}$ are both in S, a.e. We first show that $P_0 \in S$ a.e. This involves a construction that will be used a second time to show also that $P_0^c \in S$, a.e. Let $N_0 = \{g_0 < 0\}$.

Suppose that we have inductively chosen $g_1, g_2, ..., g_k \in G$ so that for $1 \leqslant j \leqslant k$,

$$\mu\left(\text{supp } g_j \cap \left[\bigcap_{i=0}^{j-1} Z(g_i)\right]\right) > 0,$$

and that we have defined

$$P_j = \{g_j > 0\} \cap \left[\bigcap_{i=0}^{j-1} Z(g_i)\right], \qquad \text{and} \qquad N_j = \{g_j < 0\} \cap \left[\bigcap_{i=0}^{j-1} Z(g_i)\right].$$

Notice that the $2j+2$ sets $\{P_i\}_{i=0}^{j}$ and $\{N_i\}_{i=0}^{j}$ are disjoint. In particular we have the following, which we use later:

$$\left(\bigcup_{i=0}^{j} P_i\right)^c = \left(\bigcup_{i=0}^{j} N_i\right) \cup \left(\bigcap_{i=0}^{j} Z(g_i)\right).$$

If $\mu(\bigcap_{i=0}^{k} Z(g_i)) \neq 0$, we choose g_{k+1} so that

$$\mu\left(\operatorname{supp} g_{k+1} \cap \left[\bigcap_{i=0}^{k} Z(g_i)\right]\right) > 0,$$

and define P_{k+1} and N_{k+1} as above.

Since $\operatorname{supp} G = X$, this recursive process terminates for some w ($\leqslant n$) steps when $\mu(\bigcap_{i=0}^{w} Z(g_i)) = 0$. If both of the sets $\bigcup_{i=0}^{w} P_i$, and $P_0 \cup [\bigcup_{i=1}^{w} N_i]$ are in S, a.e., then their intersection—which is P_0—would also be in S, a.e.

The following sublemma shows that both these sets are in S. We apply the sublemma once to the functions $g_0, g_1, g_2, ..., g_w$, and then a second time to $g_0, -g_1, -g_2, ..., -g_w$.

SUBLEMMA. *If g_i, P_i, and N_i, are defined as above, then $K = \bigcup_{i=0}^{w} P_i \in S$, a.e.*

Proof of the Sublemma. By hypothesis there is an $A \in S$, a.e. such that $\int_X g I_K \, d\mu = \int_X g I_A \, d\mu$ for all $g \in G$. To prove that $K = A$, a.e., we will show that for each $i = 0, 1, ..., w$, (i) $P_i \subseteq A$, a.e., and (ii) $A \cap N_i = \varnothing$, a.e. This is done inductively. The first induction step is included when $k = 0$.

Suppose (i) and (ii) are true for each $i < k$ where $0 \leqslant k < w$. Now X is the union of the five disjoint sets,

$$\bigcup_{i=0}^{k-1} P_i, \qquad \bigcup_{i=0}^{k-1} N_i, \qquad P_k, \qquad N_k, \qquad \bigcap_{i=0}^{k} Z(g_i).$$

We rewrite each integral in the equation

$$\int_X g_k I_A \, d\mu = \int_X g_k I_K \, d\mu$$

as the sum of five integrals over these sets.

From the definition of K and from the induction assumption (i), we have that $\bigcup_{i=0}^{k-1} P_i$ is contained, a.e., in both K and in A. Therefore $g_k I_K = g_k = g_k I_A$ a.e. on the first set. Also $g_k I_K$ vanishes a.e. on the second, fourth, and fifth set, and $g_k I_A$ vanishes a.e. on the second and fifth set. Hence we get,

$$\int_{P_k \cup N_k} g_k I_A \, d\mu = \int_{P_k} g_k I_K \, d\mu = \int_{P_k} g_k^+ \, d\mu = \int_{P_k \cup N_k} g_k^+ \, d\mu.$$

But for any $B \in \Sigma$ and any integrable function h, $\int_B h I_A \, d\mu = \int_B h^+ \, d\mu$, implies that:

$$\text{(a)} \quad B \cap \{h > 0\} \subseteq A, \text{ a.e.,} \qquad \text{and} \qquad \text{(b)} \quad A \cap [B \cap \{h < 0\}] = \varnothing, \text{ a.e.}$$

In our setting (a) and (b) become

$$\text{(a}'\text{)} \quad P_k \subseteq A, \quad \text{a.e.,} \qquad \text{and} \qquad \text{(b}'\text{)} \quad A \cap N_k = \varnothing, \quad \text{a.e.}$$

Since (a′) and (b′) are true for all k, and using the fact that $\mu(\bigcap_{k=0}^{w} Z(g_k)) = 0$, we have that

$$K = \bigcup_{k=0}^{w} P_k \subseteq A, \quad \text{a.e.,} \qquad \text{and}$$

$$A \cap \left(\bigcup_{k=0}^{w} N_k \right) = A \cap \left(\bigcup_{k=0}^{w} N_k \right) \cup \left(\bigcap_{k=0}^{w} Z(g_k) \right) = A \cap \left(\bigcup_{k=0}^{w} P_k \right)^c = \varnothing, \quad \text{a.e.}$$

Hence $K = A$, a.e., and the proof of the sublemma is completed. ∎

We have shown that for every $g \in G$, $\{g > 0\} \in S$, a.e.

The second part of the proof of the lemma is to show that for $g \in G$, $\{g \leqslant 0\} \in S$, a.e. We will apply the sublemma above.

Let $g_0 = -g$. So, $\{g \leqslant 0\} = \{g_0 > 0\} \cup Z(g_0)$. We repeat the construction preceding the sublemma. We put $P_0 = \{g_0 > 0\}$. We choose $g_1, g_2, ..., g_k \in G$, and $w < n$ so that: for $j \leqslant k$, $\mu(\text{supp } g_j \cap [\bigcap_{i=0}^{j-1} Z(g_i)]) > 0$, and $\mu(\bigcap_{i=0}^{w} Z(g_i)) = 0$. We define P_i and N_i as in the first part of the proof.

The two sets $\bigcup_{i=0}^{w} P_i$, and $P_0 \cup [\bigcup_{i=1}^{w} N_i]$ union to $\{g_0 > 0\} \cup Z(g_0)$, a.e. Hence, if both of these sets are in S, their union—which is also equal $\{g \leqslant 0\}$—is also in S. The sublemma shows that both sets are in S. We apply the sublemma once to the functions $g_0, g_1, g_2, ..., g_w$, and then a second time to $g_0, -g_1, -g_2, ..., -g_w$.

We now have shown that for all $g \in G$ both $\{g > 0\}$ and $\{g \leqslant 0\}$ are in S. Since S is closed under finite intersections and unions we have that G is S-supported ∎

COROLLARY 5.2. *Let (X, Σ, μ) be a finite, nonatomic, measure space. Let G be a finite dimensional subspace of $L_1(X, \Sigma, \mu)$ such that* supp $G = X$, *a.e.*

In the following settings properties (1), (2), and (3) below are equivalent.

(a) (X, Σ, μ) is a Baire measure space, and $S = \{\text{supp } f : f \in C^+(X)\}$;

(b) S is the subsets of X that are both open a.e. and closed a.e., and Σ is the σ-field generated by S;

(c) Suppose that the subsets of X that are both open a.e. and closed a.e. form a basis for the topology on X. Then let S be the sets are both open a.e. and closed a.e., and let Σ the Borel sets.

(1) $Q = \{M(s) : s \in \text{ext } B(L_\infty)$, and s is S-supported$\}$,

(2) $Q^+ = \{M(I_A) : A \in S\}$, and

(3) G is S-supported.

Proof. Theorem 5.1 shows that (2) implies (3), and Theorem 4.1 shows that (3) implies (1).

So suppose that (1) is true, and that $h \in L_\infty$, and $0 \leqslant h \leqslant 1$. There is an S-supported s in ext $B(L_\infty)$ such that for all $g \in G$.

$$\int_X [2h-1]\, g\, d\mu = \int_X sg\, d\mu.$$

So then

$$\int_X hg\, d\mu = \int_X \tfrac{1}{2}[1+s]\, g\, d\mu.$$

Since $\frac{1}{2}[1+s]$ is the indicator function for $\{s > 0\} \in S$, we conclude that (2) is true. ▮

Example. There is a lower semi-continuous function, f, on the real unit interval, whose level sets have Lebesgue measure zero, but such that f is not supported by the sets S that are both open a.e. and closed a.e.

To construct such an f, let $\{r_i\}$ be a listing of the rational numbers in $[0, 1]$. Let $U = \bigcup_{i=1}^{\infty} \{x : |x - r_i| < 1/2^{i+1}\}$. Let $f(x) = x$ for $x \in U$, and let $f(x) = x - 1$ for $x \notin U$. Then $f^{-1}(a)$ contains at most one point for all $-1 \leqslant a \leqslant 1$.

To show that f is lower semicontinuous let a be a real number. We need to show that $U_a = f^{-1}(a, \infty)$ is open. If $a \geqslant 0$ then $U_a = U \cap (a, \infty)$. Since U is open, U_a is open. If $a \leqslant 0$, then $f^{-1}((-\infty, a]) = U^c \cap (-\infty, a+1]$, and so is closed, Hence its complement, U_a is open.

The set $U = \text{supp } f^+$ is open. We want to observe that it is not closed a.e. We need to show that there do not exist sets V and W of zero measure such that $(U - V) \cup W$ is closed. Suppose there were such sets. We will contridict the conclusion that the measure of $(U - V) \cup W$ is equal that of

U, and that the measure of U is less than $\frac{1}{2}$. Since V has zero measure, V can not contain a sphere around any of the points r_i. Hence for each i, $U - V$ contains a sequence of points converging to r_i. It follows that the closure of $U - V$ is all of $[0, 1]$. But the closure of $U - V$ must be contained in $(U - V) \cup W$.

REFERENCES

1. D. Blackwell, The range of certain vector integrals, *Proc. Amer. Math. Soc.* **2** (1951), 390–405.
2. H. Hahn and A. Rosenthal, "Set Functions," Univ. of New Mexico Press, Albuquerque, NM, 1948.
3. H. Halkin, Liapounov's theorem on the range of a vector valued measure and Pontryagin's maximum principle, *Ark. Rational Mech.* **10** (1962), 296–304.
4. E. Hewitt, On two problems of Urysohn, *Ann. of Math.* **47** (1946), 503–509.
5. H. G. Kellerer, A topological version of Liapunov's Theorem, *Arch. Math.* **72** (1999), 206–213.
6. A. Liapounoff, Sur les functions-vecteurs complements additives, *Bull. Acad. Sci. USSR* **4** (1940), 465–478.
7. J. Lindenstrauss, A short proof of Liapanov's convexity theorem, *J. Math. Mech.* **15** (1966), 971–972.
8. A. Pinkus, "On L^1-Approximation," Cambridge Univ. Press, Cambridge, UK, 1989.
9. H. Render and H. Stroetmann, Liapounov's convexity theorem for topological measures, *Arch. Math.* **67** (1996), 331–336.
10. W. Rudin, "Real and Complex Analysis," McGraw–Hill, New York, 1974.
11. W. Rudin, "Functional Analysis," McGraw–Hill, New York, 1973.
12. D. E. Wulbert, Liapanov's and related theorems, *Proc. Amer. Math. Soc.* **108** (1990), 955–960.
13. D. E. Wulbert, Annihilating a subspace of L_1 with the sign of a continuous function, *Proc. Amer. Math. Soc.* **128** (2000), 2431–2438.
14. D. E. Wulbert, A general Hobby–Rice theorem and cake slicing, *Israel J. Math.*, in press.